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ON ROTATING POLAR-ORTHOTROPIC CIRCULAR DISKS

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Abstract—A closed-type solution is presented for a mathematical singularity which appears in Glushkov's solution of rotating polar orthotropic disks. It is further shown that optimization for stresses of such disks requires the establishment of a suitable criterion for each class of boundary conditions considered. The effect of angular acceleration on the design of rotating composite disks is indicated. Various examples are presented and discussed.

NOTATION

$A_{ij}, A_{rr}, A_{r\theta}, A_{\theta\theta}$ elastic stiffness coefficients, eqn (10)
A_s elastic shear coefficient, eqn (32)
a inner radius of annular disk
B_1, B_2 constants
b outer radius of annular disk
C_1, C_2 constants
$D_{ij}, D_{rr}, D_{\theta\theta}$ bending rigidities, eqn (A3)
$E_{ij}, E_{rr}, E_{r\theta}, E_{\theta\theta}$ elastic moduli
G elastic shear modulus
H_1, H_2 coefficients, eqns (16)–(17)
h disk thickness
K_1, K_2 constants
k orthotropy parameter, eqns (13), (A2)
$N_r, N_\theta, N_{r\theta}$ stress resultants, eqns (6), (29)
p pressure
R mass density per unit surface area, eqn (7)
r radial coordinate
u radial displacement
v circumferential displacement
w transverse deflection
z thickness coordinate
α angular acceleration
$\gamma_{r\theta}$ shear strain
$\epsilon_r, \epsilon_\theta$ normal strains
ω angular velocity
ρ mass density
σ_r, σ_θ normal stresses
$\tau_{r\theta}$ shear stress

INTRODUCTION

Thin elastic circular and annular disks are commonly used in many engineering structures and machines. The classical analysis of isotropic circular disks, including rotating disks, can be found, e.g. in Timoshenko and Goodier's monograph[1] where references to the original works of E. Winkler, C. Chree and J. H. Michell are given. The interest in polar-orthotropic disks arose in view of the extended applications of wood, composite materials and ring and rib stiffened structures in a variety of aeronautical and mechanical applications. Formulations of plane problems for polar-orthotropic disks were given, e.g. by Lekhnitskii[2], Okubo[3] and Carrier[4]. Rotating free polar-orthotropic disks were analyzed by Glushkov[5] and his solution is shown in Likhnitskii's monograph[6]. A most detailed survey on non-homogeneous polar-orthotropic circular disks of varying thickness was given by Bert[7]. More recently, the design of polar-orthotropic rotating disks with free boundaries was treated by Leissa and Vagins[8], whereas the optimization of a multi-ring composite flywheel was studied by Danfelt, Hewes and Chou[9].

In the present study a *general* closed type solution is shown for the mathematical singularity which appears in Glushkov's solution[5]. This singularity was treated by Tang[10] and Morganthaler and Bonk[11] using L'Hospital's rule. It is noted that the approach is to be separately applied to each class of boundary conditions. Various solutions are shown which enable some insight into the optimal design of rotating anisotropic disks.

1. ANALYSIS

Consider an annular disk symmetrically laminated of polar orthotropic composite layers, rotating with constant angualr velocity ω.

Hooke's law for each layer is written in terms of the elastic stiffness moduli E_{ij}

$$\sigma_r = E_{rr}\epsilon_r + E_{r\theta}\epsilon_\theta \tag{1}$$

$$\sigma_\theta = E_{r\theta}\epsilon_r + E_{\theta\theta}\epsilon_\theta \tag{2}$$

where for the considered disk

$$E_{ij}(-z) = E_{ij}(z) \qquad (i,j = r,\theta) \tag{3}$$

and the density

$$\rho(-z) = \rho(z) \tag{4}$$

is also symmetric with respect to the mid reference plane ($z = 0$).

Since no bending occurs, the single equation of motion of the disk takes the form

$$\frac{1}{r}(rN_r)_{,r} - \frac{1}{r}N_\theta = -R\omega^2 r \tag{5}$$

where the resultant forces are

$$(N_r, N_\theta) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\mathrm{d}z \tag{6}$$

and the density per unit surface area is

$$R = \int_{-h/2}^{h/2} \rho \mathrm{d}z. \tag{7}$$

The disk stress-strain relations are

$$N_r = A_{rr}\epsilon_r + A_{r\theta}\epsilon_\theta \tag{8}$$

$$N_\theta = A_{r\theta}\epsilon_r + A_{\theta\theta}\epsilon_\theta \tag{9}$$

after using eqns (1), (2) and (6). The disk's in-plane stiffnesses are defined as follows

$$A_{ij} = \int_{-h/2}^{h/2} E_{ij}\mathrm{d}z \qquad (i,j = r, \theta) \tag{10}$$

whereas the strain-displacement relations for the axisymmetric state of deformation reads

$$\epsilon_r = u_{,r} \qquad \epsilon_\theta = u/r. \tag{11}$$

The equation of motion (5) becomes a second order differential equation in terms of the radial displacement $u(r)$,

$$r^2 u_{,rr} + ru_{,r} - k^2 u = -\frac{R\omega^2}{A_{rr}} r^3 \tag{12}$$

where the orthotropy parameter

$$k = \sqrt{(A_{\theta\theta}/A_{rr})} \tag{13}$$

is unity for a symmetrically laminated *isotropic* disk.

The solution, given by Lekhnitskii[2], of eqn (12)

$$u(r) = C_1 r^k + C_2 r^{-k} + \frac{R\omega^2}{A_{rr}(k^2-9)} r^3 \tag{14}$$

does not treat the inherent singularity for $k = 3$.

2. SOLUTION FOR $k = 3$

A general solution of eqn (12), including the case $k = 3$, is obtained by using the method of variation of parameters in the form

$$u(r) = H_1(r) r^k + H_2(r) r^{-k} \tag{15}$$

where the functions $H_1(r)$ and $H_2(r)$ are as follows:

$$H_1(r) = C_1 - \frac{R\omega^2}{2kA_{rr}} \cdot \begin{cases} \dfrac{r^{3-k}}{3-k} & k \neq 3 \quad (16.1) \\ \ln r & k = 3 \quad (16.2) \end{cases}$$

$$H_2(r) = C_2 + \frac{R\omega^2}{2kA_{rr}} \cdot \begin{cases} \dfrac{r^{3+k}}{3+k} & k \neq 3 \quad (17.1) \\ \dfrac{1}{6} r^6 & k = 3. \quad (17.2) \end{cases}$$

Note that the solution for $k \neq 3$ coincides with that of eqn (14) and includes the isotropic composite disk as a special case.

The constants C_1 and C_2 in eqns (16), (17) are determined from the boundary conditions of the rotating disk whereas the stresses at each layer are obtained from eqns (1), (2), (11), (15)–(17).

3. EXAMPLE

Let us consider a clamped disk with a central hole of radius a and an outer radius b. The boundary conditions are

$$u = 0 \text{ at } r = a, b. \tag{18}$$

The obtained radial displacement is then

$$u(r) = \frac{R\omega^2 b^3}{A_{rr}} \cdot \frac{1}{9-k^2} \left\{ \frac{\left(\frac{r}{a}\right)^k - \left(\frac{a}{r}\right)^k + \left(\frac{a}{b}\right)^3 \left[\left(\frac{b}{r}\right)^k - \left(\frac{r}{b}\right)^k\right]}{\left(\frac{b}{a}\right)^k - \left(\frac{a}{b}\right)^k} - \left(\frac{r}{b}\right)^3 \right. \tag{19.1}$$

for $k \neq 3$, or

$$u(r) = \frac{R\omega^2 b^3}{A_{rr}} \cdot \frac{1}{6} \frac{\left(\frac{r}{a}\right)^3 \ln\frac{b}{r} + \left(\frac{ar}{b^2}\right)^3 \ln\frac{r}{a} - \left(\frac{a}{r}\right)^3 \ln\frac{b}{a}}{\left(\frac{b}{a}\right)^3 - \left(\frac{a}{b}\right)^3} \tag{19.2}$$

for $k = 3$. Equation (19.2), which is the solution according to eqns (16.2) and (17.2), can also be obtained by a limit procedure of eqn (19.1) for $k = 3$. This is an obviously expected result, from a physical point of view and it is true for different boundary conditions, too. A typical solution of $u(r)$ vs k is shown in Fig. 1. The numerical magnitude for $k = 3$ was calculated by using eqn (19.2) while eqn (19.1) was used in the range $k \neq 3$.

4. A VIEW TOWARDS OPTIMIZATION

The radial and the circumferential stresses (σ_r and σ_θ, respectively) in the rotating polar-orthotropic disk vary with the orthotropy parameter k.

From eqns (1), (2), (11), (15)–(17), for $k \neq 3$

$$\sigma_r = C_1(kE_{rr} + E_{r\theta})r^{k-1} - C_2(kE_{rr} - E_{r\theta})r^{-k-1} + \frac{R\omega^2}{k^2-9}\left(3 + \frac{E_{r\theta}}{E_{rr}}\right)r^2 \tag{20.1}$$

$$\sigma_\theta = C_1(kE_{r\theta} + E_{\theta\theta})r^{k-1} - C_2(kE_{r\theta} - E_{\theta\theta})r^{-k-1} + \frac{R\omega^2}{k^2-9}\left(3\frac{E_{r\theta}}{E_{rr}} + k^2\right)r^2 \tag{21.1}$$

whereas for $k = 3$

$$\sigma_r = C_1(3E_{rr} + E_{r\theta})r^2 - C_2(3E_{rr} - E_{r\theta})r^{-4} - \frac{R\omega^2}{6}\left[\ln r\left(3 + \frac{E_{r\theta}}{E_{rr}}\right) + \frac{1}{6}\left(3 - \frac{E_{r\theta}}{E_{rr}}\right)\right]r^2 \tag{20.2}$$

$$\sigma_\theta = C_1(3E_{r\theta} + E_{\theta\theta})r^2 - C_2(3E_{r\theta} - E_{\theta\theta})r^{-4} - \frac{R\omega^2}{6}\left[3\ln r\left(3 + \frac{E_{r\theta}}{E_{rr}}\right) - \frac{1}{2}\left(3 - \frac{E_{r\theta}}{E_{rr}}\right)\right]r^2. \tag{21.2}$$

The radial distribution of the stresses is dependent on the boundary conditions of the rotating disk; consequently, different criteria of optimization will be necessary for their design. The solutions for three different types of boundary conditions are presented and suitable concepts of optimization are discussed.

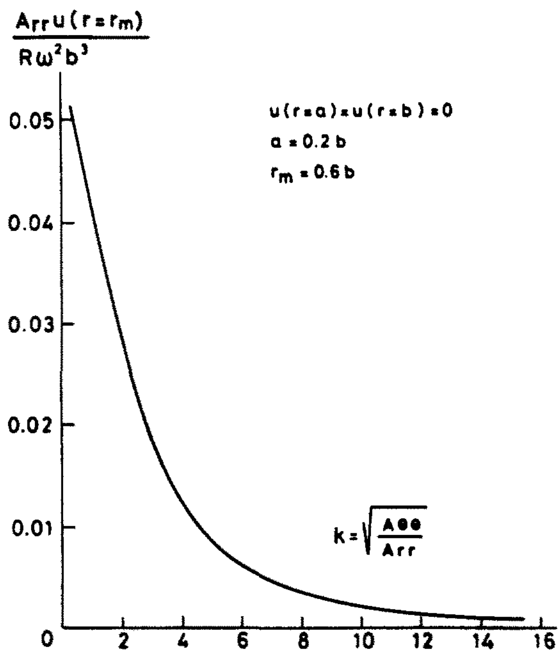

Fig. 1. Variation of non-dimensional radial displacement (at $r = r_m$) vs k.

(i) *Clamped disk*

For the clamped disk of Section 3, the radial distribution of the stresses in a homogeneous disk is shown in Fig. 2 for $k = 3$ and $a/b = 0.2$. It is noted that near the outer circumference ($r = b$) compressive stresses occur. The maximum absolute values of the hoop stresses (σ_θ) are greater than those of the radial stresses (σ_r). This fact is not true for every k. The maximum tension (σ_{max}) and the maximum compression (σ_{min}) are shown in Fig. 3 vs k. For small k, like for an isotropic disk, the radial stresses are larger than the circumferential ones.

A reasonable criterion of optimization is that defined by the requirement for minimum stress. For the case of $a/b = 0.2$, $E_{\theta\theta} \simeq 3.25\, E_{rr}$ an upper bound of $|\sigma|_{max} \simeq 0.2\,\rho\omega^2 b^2$ is found, as shown in Fig. 3 for $k \simeq 1.8$. Higher values of k yield larger hoop stresses, while smaller ones yield larger radial stresses.

A plot like that of Fig. 3 is to be prepared for any aspect ratio a/b if an optimization for k is desired.

(ii) *Free disk*

For a free disk the following boundary conditions

$$\sigma_r = 0 \text{ at } r = a,b \tag{22}$$

are to be satisfied. As a result, the coefficients C_1 and C_2 in eqns (16), (17), (20) and (21) become:

$$C_1 = \frac{R\omega^2}{(9-k^2)A_{rr}} \cdot \frac{3A_{rr} + A_{r\theta}}{kA_{rr} + A_{r\theta}} \cdot \frac{b^{3+k} - a^{3+k}}{b^{2k} - a^{2k}} \quad ,k \neq 3 \tag{23.1}$$

or

$$C_1 = \frac{R\omega^2}{6A_{rr}}\left[\frac{b^6 \ln b - a^6 \ln a}{b^6 - a^6} + \frac{1}{6}\cdot\frac{3A_{rr} - A_{r\theta}}{3A_{rr} + A_{r\theta}}\right], \quad k = 3 \tag{23.2}$$

and

$$C_2 = -\frac{R\omega^2}{(9-k^2)A_{rr}} \cdot \frac{3A_{rr} + A_{r\theta}}{kA_{rr} - A_{r\theta}} \cdot \frac{b^{3-k} - a^{3-k}}{b^{-2k} - a^{-2k}} \quad ,k \neq 3 \tag{24.1}$$

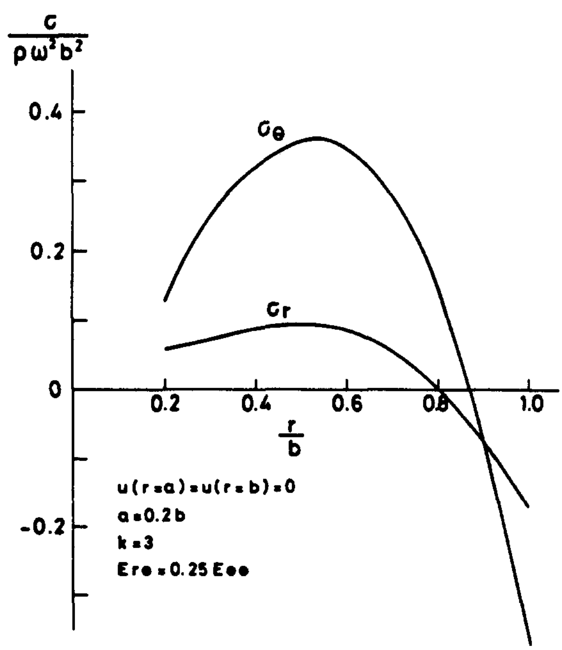

Fig. 2. Radiation variation of σ_r, σ_θ in a clamped rotating homogeneous disk.

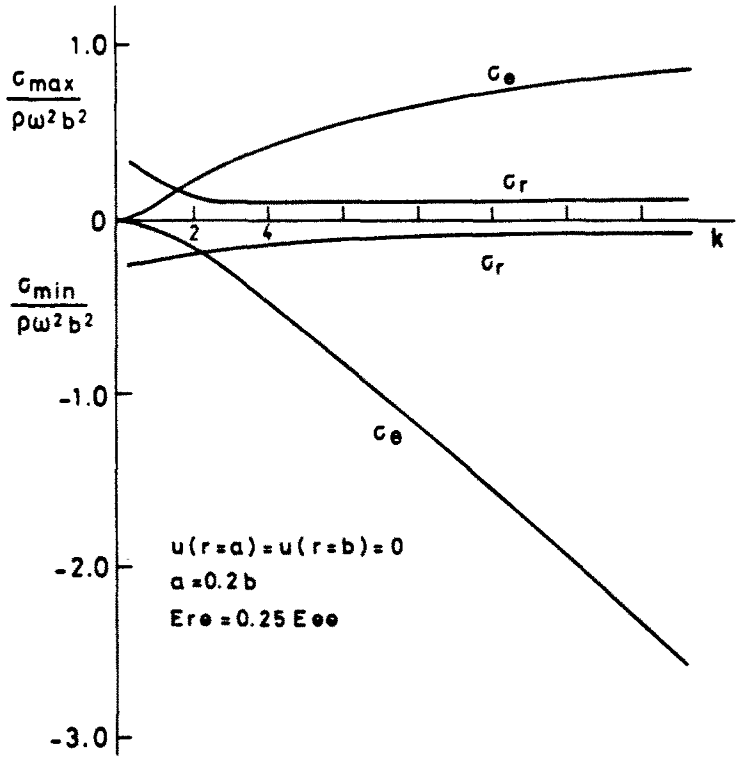

Fig. 3. Variation of extremal stresses vs k.

or

$$C_2 = -\frac{R\omega^2}{6A_{rr}} \cdot \frac{3A_{rr} + A_{r\theta}}{3A_{rr} - A_{r\theta}} \cdot \frac{\ln \frac{b}{a}}{b^{-6} - a^{-6}} \quad ,k = 3. \tag{24.2}$$

The distribution of the stresses is presented in Fig. 4, for $k = 3$. The hoop stresses are significantly above the radial ones—a result which is also true for isotropic disks of the same boundary conditions. Compressive stresses do not exist. However, for large orthotropy parameters ($k > 5$) compression may appear.

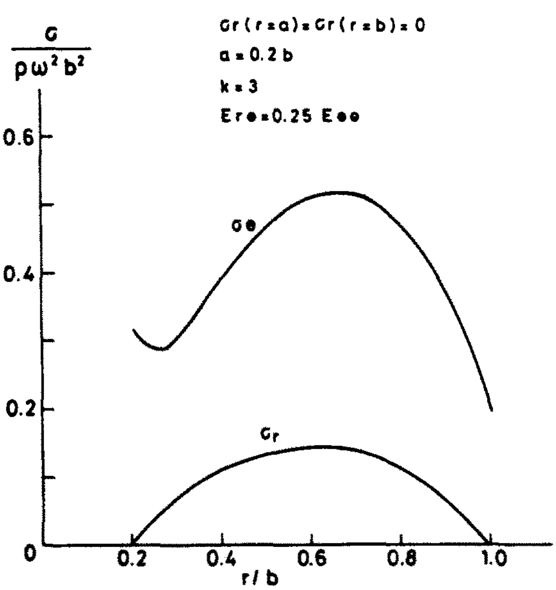

Fig. 4. Radial variation of σ_r, σ_θ in a free rotating one layer disk.

The fact that the maximum circumferential stress ($\sigma_{\theta_{max}}$) is always larger than the maximum radial stress ($\sigma_{r_{max}}$) justifies an optimization criterion of choosing a parameter k which will yield a minimum magnitude of $\sigma_{\theta_{max}}$. This criterion was employed by Leissa and Vagins[8], but it should be emphasized that it is only useful for disks with free boundaries, eqn (22).

The effect of different aspect ratio (a/b) on the maximum hoop stress ($\sigma_{\theta_{max}}$) in isotropic disks ($k = 1$) is very small. But, it can be seen in Fig. 5 that for $k \neq 1$ an optimization of the aspect ratio is of great importance. For example, for $k = 2$, $\sigma_{\theta_{max}}$ in a disk of $a/b = 0.4$ is about 60% larger than that in a disk of $a/b = 0.2$. Figure 5 proposes therefore, data for an optimal design of the orthotropy parameter (k) and/or the aspect ratio (a/b), according to the requirements and the constraints of a specific design.

(iii) *Clamped-free disk*

The boundary conditions of a rotating disk with a free outer boundary ($r = b$) and a relatively rigid shaft in the central hole of radius a, can be written as

$$u(r = a) = \sigma_r(r = b) = 0. \tag{25}$$

This combination of boundary conditions yields the following expressions for the constants C_1 and C_2:

$$C_1 = \frac{R\omega^2}{(9-k^2)A_{rr}} \cdot \frac{(kA_{rr} - A_{r\theta})a^{k+3} + (3A_{rr} + A_{r\theta})b^{k+3}}{(kA_{rr} - A_{r\theta})a^{2k} + (kA_{rr} + A_{r\theta})b^{2k}} \quad ,k \neq 3 \tag{26.1}$$

or

$$C_1 = \frac{R\omega^2}{6A_{rr}} \cdot \frac{(3A_{rr} - A_{r\theta})\left[a^6 \ln a + \frac{1}{6}(b^6 - a^6)\right] + (3A_{rr} + A_{r\theta})b^6 \ln b}{(3A_{rr} - A_{r\theta})a^6 + (3A_{rr} + A_{r\theta})b^6} \quad ,k = 3 \tag{26.2}$$

and

$$C_2 = \frac{R\omega^2}{(9-k^2)A_{rr}} \cdot \frac{(kA_{rr} + A_{r\theta})a^{3-k} - (3A_{rr} + A_{r\theta})b^{3-k}}{(kA_{rr} + A_{r\theta})a^{-2k} + (kA_{rr} - A_{r\theta})b^{-2k}} \quad ,k \neq 3 \tag{27.1}$$

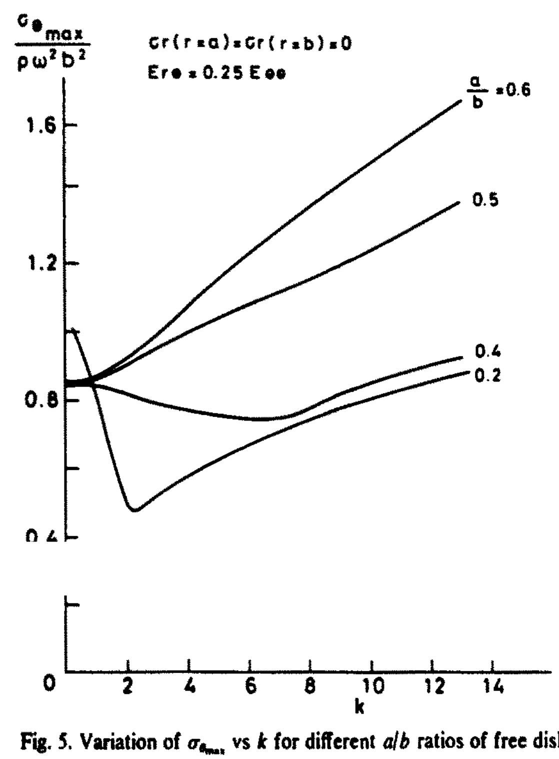

Fig. 5. Variation of $\sigma_{\theta_{max}}$ vs k for different a/b ratios of free disk.

or

$$C_2 = \frac{R\omega^2}{6A_{rr}} \cdot \frac{(3A_{rr} + A_{r\theta})\ln\frac{a}{b} - A_{rr}}{(3A_{rr} + A_{r\theta})a^{-6} + (3A_{rr} - A_{r\theta})b^{-6}} \quad ,k = 3 \tag{27.2}$$

The maximum stresses strongly depend on the aspect ratio (a/b) and the orthotropy parameter (k), as shown in Fig. 6. The maximum radial stresses are significantly large, for small k, relative to the circumferential ones, but the opposite is true for large k. The best criterion of optimization is apparently the lowest maximum stress, which is $\sigma_{r_{max}} = \sigma_{\theta_{max}}$, corresponding to an optimal k for each a/b ratio.

This criterion leads to the conclusion that as the a/b ratio increases the upper bound of the stresses is reduced for the optimal disk. The following table presents a numerical example of this conclusion which is obtained from Fig. 6:

Table 1. Variation of σ_{max} with a/b ratio for optimal k

$\frac{a}{b}$	optimal k	$\frac{\sigma_{max}}{\rho\omega^2 b^2}$
0.02	1.2	0.39
0.2	1.4	0.37
0.6	2.0	0.28

It should be noted that for $k > 5$ compressive hoop stresses appear near the outer circumference ($r = b$) but the design is still controlled by the tensile stresses in case the material has equal strengths in compression and tension.

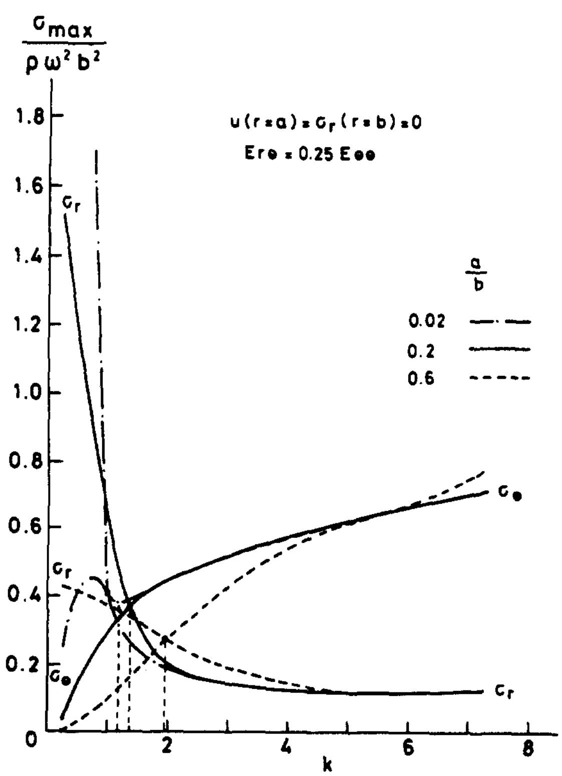

Fig. 6. Variation of σ_{max} vs k for different a/b ratios of clamped free disk.

5. EFFECT OF ANGULAR ACCELERATION

When an angular acceleration (α) of the disk is concerned, a circumferential equation of motion

$$N_{r\theta,r}+\frac{2}{r}N_{r\theta}=R\alpha r \tag{28}$$

complements the radial equation of motion (12), where the inplane shear force is defined by

$$N_{r\theta}=\int_{-h/2}^{h/2}\tau_{r\theta}\mathrm{d}z. \tag{29}$$

At each material point (r, θ, z) the shear stress is given by

$$\tau_{r\theta}(r,z)=G(z)\gamma_{r\theta}(r) \tag{30}$$

and for the laminated disk a similar relation holds for the resultant shear force:

$$N_{r\theta}(r)=A_s\gamma_{r\theta}(r) \tag{31}$$

where

$$A_s=\int_{-h/2}^{h/2}G(z)\mathrm{d}z \tag{32}$$

and

$$\gamma_{r\theta}=v_{,r}-\frac{v}{r} \tag{33}$$

for the axisymmetric state of displacement considered.

Substitution of eqns (31) and (33) into eqn (28) gives the displacement equation of motion for v:

$$v_{,rr}+\frac{1}{r}v_{,r}-\frac{v}{r^2}=\frac{R\alpha}{A_s}r \tag{34}$$

with the solution

$$v(r)=B_1r+B_2r^{-1}+\frac{R\alpha}{8A_s}r^3 \tag{35}$$

for the circumferential displacement. The constant coefficients B_1 and B_2 are determined by the boundary conditions. Considering, for example, a disk with a central rigid shaft, one writes

$$v(r=a)=\tau_{r\theta}(r=b)=0 \tag{36}$$

with the solution

$$v(r)=\frac{R\alpha}{8A_s}a^2\left\{\frac{r^3}{a^2}-\left[1+\left(\frac{b}{a}\right)^4\right]r+\frac{b^4}{a^2r}\right\}. \tag{37}$$

It should be pointed out that the solution of $v(r)$ for the laminated orthotropic disk is essentially the same as for the isotropic case, except for the thickness variation of $\tau_{r\theta}$, which is

$$\tau_{r\theta}(r,z)=G(z)\frac{R\alpha b^2}{4A_s}\left[\left(\frac{r}{b}\right)^2-\left(\frac{b}{a}\right)^2\right]. \tag{38}$$

6. CONCLUSIONS

A closed form solution is presented for a singularity which seems to appear in the "classical" solution of rotating orthotropic disks for $A_{\theta\theta} = 9A_{rr}$. It is shown that the general solution coincides with results for various boundary conditions when a limit process is used for the point of singularity in the "classical" solution.

It is further shown that in order to optimize the orthotropy parameter (k) of rotating circular disks, different criteria of optimization are necessary for different boundary conditions. If, for example, minimum upper bound of the stresses is suggested for the clamped disk, a minimum magnitude of the circumferential stress (σ_θ) is a sufficient criterion for the free disk, and for the clamped-free disk the best criterion for optimization of k is $\sigma_{r_{max}} = \sigma_{\theta_{max}}$.

The optimization can be significantly improved if the design of the disk permits also an optimization of the aspect-ratio a/b. The combined optimization of k and a/b ratio may considerably reduce the extremal stresses as shown, e.g. in Fig. 5 for a free rotating disk.

Finally, it is shown that a consideration of angular acceleration does not affect the above mentioned conclusions because the orthotropy parameter k does not appear in the shear stress term. However, in this case, shear stress and its effects on design should not be ignored.

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APPENDIX

A. *Deflection of circular plate*

The limit process can be used for a laminated polar orthotropic plate which is loaded by a uniformly distributed transverse pressure (p). It was shown (see, e.g. [6], pp. 369–373), that the deflection takes the form

$$w(r) = K_1 + K_2 r^{1+k} + \frac{p}{8(9-k^2)D_{rr}} r^4. \tag{A1}$$

The plate orthotropy parameter

$$k = \sqrt{\left(\frac{D_{\theta\theta}}{D_{rr}}\right)}. \tag{A2}$$

can take any positive value, except for 1 and 3.

The plate bending rigidities are

$$D_{ij} = \int_{-h/2}^{+h/2} E_{ij} z^2 \mathrm{d}z \quad (i,j = r,\theta). \tag{A3}$$

The same singularity, for $k = 3$, appears here as in eqn (14).

For a clamped orthotropic circular plate of radius b, with boundary conditions

$$w(r = b) = w_{,r}(r = b) = 0 \tag{A4}$$

the deflection function becomes

$$w(r) = \frac{pb^4}{8(9-k^2)(1+k)D_{rr}}\left[3-k-4\left(\frac{r}{b}\right)^{1+k}+(1+k)\left(\frac{r}{b}\right)^4\right] \quad \text{(A5)}$$

This solution is replaced for $k = 3$ by the following expression

$$w(r) = \frac{pb^4}{192D_{rr}}\left\{1-\left(\frac{r}{b}\right)^4\left[1-\ln\left(\frac{r}{b}\right)^4\right]\right\} \quad \text{(A6)}$$

It is noted that although the singularity of eqn (A5) for $k = 3$, disappears at $r = 0$, and so the maximum deflection

$$w_{max} = w(r=0) = \frac{pb^4}{192D_{rr}} \quad \text{(A7)}$$

is correctly obtained—only eqn (A6) gives the deflection function for the entire region.